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ScienceDirect
Journal of Geometry and Physics 57 (2007) 2539–2546
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Special tensors in the deformation theory of quadratic algebras for the classical Lie algebras

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Received 15 February 2007; accepted 12 September 2007
Available online 20 September 2007

Abstract

Using deformation theory, Braverman and Joseph constructed certain primitive ideals in the enveloping algebras of the simple Lie algebras. Except in the case $\mathfrak{sl}(2,\mathbb{C})$, there is a special value of the deformation parameter giving an ideal of infinite codimension. For the classical Lie algebras, the uniqueness of the special value is equivalent to the existence of tensors with very particular properties. The existence of these tensors was concluded abstractly by Braverman and Joseph but here we present explicit formulae. This allows a rather direct computation of the special value of the deformation parameter.

JGP subject classifications: Lie groups and Lie (super)algebras

MSC: primary 17B35; secondary 17A45; 17B81; 58J70

Keywords: Special tensors; Joseph ideal; Deformation; Lie algebra; Quadratic algebra

1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra. Let $\mathfrak{g} \circledcirc \mathfrak{g}$ denote the Cartan product of $\mathfrak{g}$ with itself, namely the unique irreducible component of $\mathfrak{g} \otimes \mathfrak{g}$ whose highest weight is twice the highest weight of $\mathfrak{g}$ itself. Also, let us denote by

$$\mathfrak{g} \otimes \mathfrak{g} \ni X \otimes Y \longmapsto X \circledcirc Y \in \mathfrak{g} \circledcirc \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$$

the invariant projection onto the Cartan product. Let $[X, Y]$ denote the Lie bracket and $\langle X, Y\rangle$ the Killing form on $\mathfrak{g}$. Then in the full tensor algebra $\bigotimes \mathfrak{g}$ let us consider the two-sided ideal I_λ generated by elements of the form

$$X \otimes Y - X \circledcirc Y - \frac{1}{2}[X, Y] - \lambda\langle X, Y\rangle \in \bigotimes^2 \mathfrak{g} \oplus \mathfrak{g} \oplus \mathbb{C}, \quad \forall X, Y \in \mathfrak{g}. \tag{1.1}$$

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.09.004

Let us denote by A_λ the quotient algebra $\bigotimes \mathfrak{g}/I_\lambda$. The following theorem was proved by Braverman and Joseph [2] based on earlier work of Joseph [7] and Braverman and Gaitsgory [1].

Theorem 1.1. *For each complex simple Lie algebra not isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, there is precisely one value of λ for which A_λ is infinite-dimensional.*

The relations (1.1) imply that the ideals I_λ descend to corresponding ideals $\bar{I}_\lambda$ in $\mathfrak{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. In this context, with the special linear series of Lie algebras excluded, the ideals $\bar{I}_\lambda$ are the Joseph ideals [7]. To prove that, for all λ save for a special value, the ideal is of finite codimension, Braverman and Joseph [2] present an abstract argument, which they make explicit for the symplectic and special linear algebras. They remark, however, that 'in general such verification seems very difficult'. This is what we accomplish for the orthogonal algebras. For completeness and convenience, we also present the special tensors that may be used in direct proofs of the symplectic and special linear cases.

In Sections 2–4 we present the tensors that enable us to list, in Theorem 5.5, the special values of λ in order that A_λ might be infinite-dimensional. That A_λ is, indeed, infinite-dimensional for these special values can sometimes be established by explicit construction. This was done in [3] for the orthogonal algebras and in Section 5 we do it for the special linear series. Braverman and Joseph [2] instead base the existence of these infinite-dimensional algebras upon [1].

The first author would like to thank Nolan Wallach for drawing his attention to the Joseph ideal in response to a talk given at the University of California, San Diego, on higher symmetries of the Laplacian [3].

2. The orthogonal case

We shall use index conventions for tensors as is standard in differential geometry. More precisely, we use the abstract index notation of Penrose [8]. For example V^a will denote an element in $\mathbb{C}^n$ viewed as the defining representation of $\mathfrak{so}(n,\mathbb{C})$. A skew tensor, i.e. an element of $\Lambda^2\mathbb{C}^n$, will be denoted by V^{ab} such that $V^{ab} = -V^{ba}$. Also g_{ab} will denote the non-degenerate quadratic form preserved by $\mathfrak{so}(n,\mathbb{C})$ and g^{ab} its inverse. We shall 'raise and lower' indices without comment: so $X^a = g^{ab}X_b$ and $X_a = g_{ab}X^b$ where a repeated index denotes the invariant pairing between vectors and covectors. The mapping $X^a \mapsto X_a$ is just the canonical isomorphism between the defining representation and its dual. Finally, the adjoint representation on $\mathfrak{so}(n,\mathbb{C})$ is realised as $V^a{}_b$ where V^{ab} is skew.

Theorem 2.1. *For $\lambda \neq -\frac{n-4}{4(n-1)(n-2)}$ and $n \geq 5$, the two-sided ideal in $\bigotimes \mathfrak{so}(n,\mathbb{C})$ generated by*

$$X \otimes Y - X \circledcirc Y - \frac{1}{2}[X, Y] - \lambda\langle X, Y\rangle, \quad \textit{for } X, Y \in \mathfrak{so}(n,\mathbb{C})$$

contains $\mathfrak{so}(n,\mathbb{C})$, the first graded piece of $\bigotimes \mathfrak{so}(n,\mathbb{C})$.

Proof. The ideal is generated by tensors of the form

$$V^{abcd} - (\circledcirc V)^{abcd} - \frac{1}{2}(V^a{}_b{}^{bd} - V^d{}_b{}^{ba}) + \lambda(n-2)V^{ab}{}_{ab} \quad \text{for } V^{abcd} = -V^{bacd} = -V^{abdc},$$

where $(\circledcirc V)^{abcd}$ denotes the Cartan part of V^{abcd}. (There is, of course, an explicit formula for $(\circledcirc V)^{abcd}$ but we shall not need all of it.) Consider the following tensor:

$$\begin{aligned} S^{abcdef} = {} & 2g^{af}g^{be}T^{cd} - 2g^{ae}g^{bf}T^{cd} - 2g^{cf}g^{de}T^{ab} + 2g^{ce}g^{df}T^{ab} + g^{ac}g^{be}T^{df} \\ & - g^{bc}g^{ae}T^{df} - g^{ad}g^{be}T^{cf} + g^{bd}g^{ae}T^{cf} - g^{ac}g^{bf}T^{de} + g^{bc}g^{af}T^{de} \\ & + g^{ad}g^{bf}T^{ce} - g^{bd}g^{af}T^{ce} - g^{ac}g^{de}T^{bf} + g^{ad}g^{ce}T^{bf} + g^{bc}g^{de}T^{af} \\ & - g^{bd}g^{ce}T^{af} + g^{ac}g^{df}T^{be} - g^{ad}g^{cf}T^{be} - g^{bc}g^{df}T^{ae} + g^{bd}g^{cf}T^{ae}, \end{aligned}$$

for $T^{ab} = -T^{ba}$. It is immediate that $S^{abcdef} = -S^{cdabef}$ and readily verified that

$$Z^{abcdef} \equiv \frac{1}{3}(S^{abcdef} + S^{abefcd}) + \frac{1}{6}(S^{abcedf} - S^{abdecf} - S^{abcfde} + S^{abdfce})$$

is given by

$$\begin{aligned}Z^{abcdef} &= 2g^{ce}g^{df}T^{ab} - 2g^{de}g^{cf}T^{ab} - \frac{1}{2}g^{ac}g^{de}T^{bf} + \frac{1}{2}g^{ad}g^{ce}T^{bf} + \frac{1}{2}g^{bc}g^{de}T^{af} - \frac{1}{2}g^{bd}g^{ce}T^{af}\\ &\quad + \frac{1}{2}g^{ac}g^{df}T^{be} - \frac{1}{2}g^{ad}g^{cf}T^{be} - \frac{1}{2}g^{bc}g^{df}T^{ae} + \frac{1}{2}g^{bd}g^{cf}T^{ae} - \frac{1}{2}g^{ae}g^{cf}T^{bd} + \frac{1}{2}g^{ae}g^{df}T^{bc}\\ &\quad + \frac{1}{2}g^{be}g^{cf}T^{ad} - \frac{1}{2}g^{be}g^{df}T^{ac} + \frac{1}{2}g^{af}g^{ce}T^{bd} - \frac{1}{2}g^{af}g^{de}T^{bc} - \frac{1}{2}g^{bf}g^{ce}T^{ad} + \frac{1}{2}g^{bf}g^{de}T^{ac}.\end{aligned}$$

Generally, $S^{abcdef} \mapsto Z^{abcdef}$ is the formula for the $\mathfrak{sl}(n, \mathbb{C})$-invariant projection

$$\yng(1,1) \otimes \yng(1,1) \otimes \yng(1,1) \longrightarrow \yng(1,1) \otimes \yng(2,2).$$

In this case, however, the result is manifestly pure trace in $cdef$. It follows that under the further $\mathfrak{so}(n, \mathbb{C})$-invariant projection

$$\yng(1,1) \otimes \yng(1,1) \otimes \yng(1,1) \longrightarrow \yng(1,1) \otimes \yng(2,2) \longrightarrow \yng(1,1) \otimes \yng(2,2)_\circ$$

where $\circ$ denotes the trace-free part, the tensor S^{abcdef} maps to zero. But, for $n \geq 5$, this is the Cartan part in $cdef$. Its skew symmetry $S^{abcdef} = -S^{cdabef}$ ensures that the Cartan part is also zero with respect to $abcd$. Therefore, we may immediately reduce S^{abcdef} in two different ways with respect to the given ideal. We obtain, after a short calculation,

$$S^a{}_b{}^{bdef} = (n-4)[g^{af}T^{de} - g^{ae}T^{df} + g^{de}T^{af} - g^{df}T^{ae}],$$

which is skew in ad. Therefore $S^{ab}{}_{ab}{}^{cd} = 0$ and

$$S^{abcdef} \simeq \frac{1}{2}(S^a{}_b{}^{bdef} - S^d{}_b{}^{baef}) = (n-4)[g^{af}T^{de} - g^{ae}T^{df} + g^{de}T^{af} - g^{df}T^{ae}].$$

Tracing over de now gives

$$g^{af}T^{de} - g^{ae}T^{df} + g^{de}T^{af} - g^{df}T^{ae} \simeq (n-2)T^{af}.$$

Altogether,

$$S^{abcdef} \simeq (n-2)(n-4)T^{af}. \tag{2.1}$$

On the other hand,

$$S^{abc}{}_d{}^{df} = -g^{af}T^{bc} + g^{bf}T^{ac} - 2(n-2)g^{cf}T^{ab} - (n-3)g^{ac}T^{bf} + (n-3)g^{bc}T^{af}$$

so

$$S^{abc}{}_d{}^{df} - S^{abf}{}_d{}^{dc} = -(n-4)[g^{ac}T^{bf} - g^{bc}T^{af} - g^{af}T^{bc} + g^{bf}T^{ac}]$$

and

$$S^{abcd}{}_{cd} = 2(n-1)(n-2)T^{ab}.$$

Therefore,

$$S^{abcdef} \simeq -\frac{n-4}{2}[g^{ac}T^{bf} - g^{bc}T^{af} - g^{af}T^{bc} + g^{bf}T^{ac}] - 2\lambda(n-1)(n-2)^2T^{af}.$$

But tracing over bc gives

$$g^{ac}T^{bf} - g^{bc}T^{af} - g^{af}T^{bc} + g^{bf}T^{ac} \simeq -(n-2)T^{af}$$

and so

$$S^{abcdef} \simeq (n-2)\left[\frac{n-4}{2} - 2\lambda(n-1)(n-2)\right]T^{af}. \tag{2.2}$$

Comparing (2.1) with (2.2), we conclude that T^{ab} must be in the ideal unless we have $\lambda = -\frac{n-4}{4(n-1)(n-2)}$. This is exactly what we wanted to prove. □

3. The symplectic case

Theorem 3.1. *For $\lambda \neq -\frac{1}{16(n+1)}$ and $n \geq 2$, the two-sided ideal in $\bigotimes \mathfrak{sp}(2n, \mathbb{C})$ generated by*

$$X \otimes Y - X \circledcirc Y - \frac{1}{2}[X, Y] - \lambda\langle X, Y\rangle, \quad \text{for } X, Y \in \mathfrak{sp}(2n, \mathbb{C})$$

contains $\mathfrak{sp}(2n, \mathbb{C})$, the first graded piece of $\bigotimes \mathfrak{sp}(2n, \mathbb{C})$.

Proof. Let ω^{ab} denote the skew form preserved by $\mathfrak{sp}(2n, \mathbb{C})$ and adopt the convention that $\omega^{ac}\omega_{bc}$ is the identity. In particular $\omega^{ab}\omega_{ab} = 2n$. If we use ω_{ab} to lower indices according to $X_b = X^a\omega_{ab}$, then we may identify $\mathfrak{sp}(2n, \mathbb{C})$ as symmetric tensors $T^{ab} = T^{ab}$ and the ideal is generated by

$$V^{abcd} - (\circledcirc V)^{abcd} - \frac{1}{2}(V^{a}{}_{b}{}^{bd} + V^{d}{}_{b}{}^{ba}) + 2\lambda(n+1)V^{ab}{}_{ab} \quad \text{for } V^{abcd} = V^{bacd} = V^{abdc}.$$

Now consider the tensor

$$\begin{aligned} S^{abcdef} = {} & 4\omega^{af}\omega^{be}T^{cd} + 4\omega^{ae}\omega^{bf}T^{cd} - 4\omega^{cf}\omega^{de}T^{ab} - 4\omega^{ce}\omega^{df}T^{ab} - \omega^{ac}\omega^{be}T^{df} \\ & - \omega^{bc}\omega^{ae}T^{df} - \omega^{ad}\omega^{be}T^{cf} - \omega^{bd}\omega^{ae}T^{cf} - \omega^{ac}\omega^{bf}T^{de} - \omega^{bc}\omega^{af}T^{de} \\ & - \omega^{ad}\omega^{bf}T^{ce} - \omega^{bd}\omega^{af}T^{ce} - \omega^{ac}\omega^{de}T^{bf} - \omega^{ad}\omega^{ce}T^{bf} - \omega^{bc}\omega^{de}T^{af} \\ & - \omega^{bd}\omega^{ce}T^{af} - \omega^{ac}\omega^{df}T^{be} - \omega^{ad}\omega^{cf}T^{be} - \omega^{bc}\omega^{df}T^{ae} - \omega^{bd}\omega^{cf}T^{ae}, \end{aligned}$$

for $T^{ab} = T^{ba}$. It is immediate that $S^{abcdef} = -S^{cdabef}$ and readily verified that

$$Z^{abcdef} \equiv \frac{1}{6}(S^{abcdef} + S^{abdefd} + S^{abcfde} + S^{abfedc} + S^{abedfc} + S^{abdfec})$$

vanishes. This is already the Cartan part with respect to the $cdef$ indices. Therefore, we may reduce S^{abdcef} modulo the ideal in two different ways. We obtain

$$S^{abcdef} \simeq -2(n-1)(\omega^{ae}T^{df} + \omega^{de}T^{af} + \omega^{af}T^{de} + \omega^{df}T^{ae}) \simeq -4(n-1)(n+1)T^{af}$$

or

$$\begin{aligned} S^{abcdef} & \simeq -(n-1)(\omega^{ac}T^{bf} + \omega^{bc}T^{af} + \omega^{af}T^{bc} + \omega^{bf}T^{ac}) + 32\lambda(n-1)(n+1)^2T^{af} \\ & \simeq -2(n-1)(n+1)T^{af} + 32\lambda(n-1)(n+1)^2T^{af}. \end{aligned}$$

Comparing these two reductions, we see that T^{ab} lies in the ideal unless $\lambda = -\frac{1}{16(n+1)}$. This is what we wanted to prove. □

Note that the critical value of λ for $\mathfrak{sp}(4, \mathbb{C}) \cong \mathfrak{so}(5, \mathbb{C})$ may be computed either from Theorem 2.1 or Theorem 3.1. Its common value is $-1/48$.

4. The special linear case

Theorem 4.1. *For $\lambda \neq -\frac{1}{8(n+1)}$ and $n \geq 3$, the two-sided ideal in $\bigotimes \mathfrak{sl}(n, \mathbb{C})$ generated by*

$$X \otimes Y - X \circledcirc Y - \frac{1}{2}[X, Y] - \lambda\langle X, Y\rangle, \quad \text{for } X, Y \in \mathfrak{sl}(n, \mathbb{C})$$

contains $\mathfrak{sl}(n, \mathbb{C})$, the first graded piece of $\bigotimes \mathfrak{sl}(n, \mathbb{C})$.

Proof. If we identify $\mathfrak{sl}(n, \mathbb{C})$ with trace-free tensors $T^{a}{}_{b}$ in the usual manner, then the ideal is generated by tensors of the form

$$V^{a}{}_{b}{}^{c}{}_{d} - (\circledcirc V)^{a}{}_{b}{}^{c}{}_{d} - \frac{1}{2}(V^{a}{}_{b}{}^{b}{}_{d} - V^{b}{}_{d}{}^{a}{}_{b}) - 2\lambda n\, V^{a}{}_{b}{}^{b}{}_{a} \quad \text{for } V^{a}{}_{a}{}^{c}{}_{d} = 0 = V^{a}{}_{b}{}^{c}{}_{c}. \tag{4.1}$$

Consider the tensor

$$S^a{}_b{}^c{}_d{}^e{}_f = \delta^e{}_d\delta^c{}_f T^a{}_b - \frac{1}{n}\delta^c{}_d\delta^e{}_f T^a{}_b - \delta^e{}_b\delta^a{}_f T^c{}_d + \frac{1}{n}\delta^a{}_b\delta^e{}_f T^c{}_d$$
$$+\delta^a{}_d\delta^e{}_b T^c{}_f - \frac{1}{n}\delta^a{}_d\delta^e{}_f T^c{}_b - \delta^c{}_b\delta^e{}_d T^a{}_f + \frac{1}{n}\delta^c{}_b\delta^e{}_f T^a{}_d$$

for $T^a{}_a = 0$. It is immediate that $S^a{}_b{}^c{}_d{}^e{}_f = -S^c{}_d{}^a{}_b{}^e{}_f$. In particular, the Cartan part of $S^a{}_b{}^c{}_d{}^e{}_f$ with respect to the indices $abcd$ vanishes. Hence, we may use (4.1) to reduce $S^a{}_b{}^c{}_d{}^e{}_f$ modulo the given ideal. We obtain

$$S^a{}_b{}^c{}_d{}^e{}_f \simeq -(n-1)\delta^e{}_d T^a{}_f - \delta^a{}_f T^e{}_d + \delta^a{}_d T^e{}_f + \delta^e{}_f T^a{}_d \simeq -\frac{1}{2}n(n-2)T^a{}_f.$$

On the other hand, it is readily verified that

$$Z^a{}_b{}^c{}_d{}^e{}_f \equiv \frac{1}{4}(S^a{}_b{}^c{}_d{}^e{}_f + S^a{}_b{}^e{}_d{}^c{}_f + S^a{}_b{}^c{}_f{}^e{}_d + S^a{}_b{}^e{}_f{}^c{}_d)$$

is given by

$$Z^a{}_b{}^c{}_d{}^e{}_f = \frac{1}{2}\delta^e{}_d\delta^c{}_f T^a{}_b - \frac{1}{4}\delta^c{}_b\delta^e{}_d T^a{}_f - \frac{1}{2n}\delta^c{}_d\delta^e{}_f T^a{}_b + \frac{1}{4n}\delta^a{}_b\delta^e{}_f T^c{}_d - \frac{1}{4n}\delta^a{}_d\delta^e{}_f T^c{}_b$$
$$+\frac{1}{4n}\delta^c{}_b\delta^e{}_f T^a{}_d - \frac{1}{4}\delta^e{}_b\delta^c{}_f T^a{}_d - \frac{1}{4n}\delta^c{}_d\delta^a{}_f T^e_b + \frac{1}{4n}\delta^a{}_b\delta^e{}_d T^c{}_f + \frac{1}{4n}\delta^a{}_b\delta^c{}_f T^e{}_d$$
$$+\frac{1}{4n}\delta^c{}_d\delta^e{}_b T^a{}_f - \frac{1}{2n}\delta^c{}_f\delta^e{}_d T^a{}_b + \frac{1}{4n}\delta^a{}_b\delta^c{}_d T^e{}_f - \frac{1}{4n}\delta^a{}_f\delta^e{}_d T^c{}_b + \frac{1}{4n}\delta^c{}_b\delta^e{}_d T^a{}_f$$
$$-\frac{1}{4n}\delta^a{}_d\delta^c{}_f T^e_b + \frac{1}{4n}\delta^e{}_b\delta^c{}_f T^a{}_d + \frac{1}{2}\delta^e{}_f\delta^c{}_d T^a{}_b - \frac{1}{4}\delta^c{}_b\delta^e{}_f T^a{}_d - \frac{1}{4}\delta^e{}_b\delta^c{}_d T^a{}_f.$$

Generally, $S^a{}_b{}^c{}_d{}^e{}_f \mapsto Z^a{}_b{}^c{}_d{}^e{}_f$ followed by the removal of all traces in the $cdef$ indices is the Cartan projection in these indices. In this case, however, $Z^a{}_b{}^c{}_d{}^e{}_f$ is manifestly pure trace and so this Cartan part of $S^a{}_b{}^c{}_d{}^e{}_f$ vanishes. This allows us to use (4.1) with respect to the $cdef$ indices to conclude that

$$S^a{}_b{}^c{}_d{}^e{}_f \simeq \frac{1}{2}(\delta^c{}_f T^a{}_b - (n-1)\delta^c{}_b T^a{}_f + \delta^a{}_b T^c{}_f - \delta^a{}_f T^c{}_b) + 2\lambda n(n-2)(n+1)T^a{}_f$$
$$\simeq -\frac{1}{4}n(n-2)T^a{}_f + 2\lambda n(n-2)(n+1)T^a{}_f.$$

Comparing this with our previous reduction, we see that $T^a{}_b$ lies in the ideal unless $\lambda = -\frac{1}{8(n+1)}$. This is what we wanted to prove. □

Note that the critical value of λ for $\mathfrak{sl}(4,\mathbb{C}) \cong \mathfrak{so}(6,\mathbb{C})$ may be computed either from Theorem 2.1 or from Theorem 4.1. Its common value is $-1/40$.

5. Remarks and conclusions

We should explain how the special tensors used in the proofs of Theorems 2.1, 3.1 and 4.1 arise. For all simple Lie algebras other than the special linear series, there is a common source as follows. Let $\mathfrak{g}$ denote a complex simple Lie algebra and let Φ denote the composition

$$\Lambda^2\mathfrak{g}\otimes\mathfrak{g} \hookrightarrow \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \xrightarrow{\mathrm{Id}\otimes\_\circledcirc\_} \mathfrak{g}\otimes\circledcirc^2\mathfrak{g}.$$

Braverman and Joseph [2] observe that the following is true.

Theorem 5.1. *For any simple complex Lie algebra* $\mathfrak{g}$ *not isomorphic to* $\mathfrak{sl}(n,\mathbb{C})$*,*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \Lambda^2\mathfrak{g}\otimes\mathfrak{g}) = 2 \quad \textit{and} \quad \dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}\otimes\circledcirc^2\mathfrak{g}) = 1.$$

Proof. A case-by-case verification using, for example, Klimyk's formula. □

Corollary 5.2. *For any simple complex Lie algebra* $\mathfrak{g}$ *not isomorphic to* $\mathfrak{sl}(n, \mathbb{C})$,

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \ker \Phi) \geq 1.$$

This result is used abstractly by Braverman and Joseph [2] and our proofs are very much motivated by this approach: in proving Theorems 2.1 and 3.1 we find explicit non-zero homomorphisms $\mathfrak{g} \to \ker \Phi$. In fact, it is easily verified that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \ker \Phi)$ is one-dimensional so our homomorphisms are unique up to scale.

For the special linear algebras the dimensions are different:

Theorem 5.3. *For* $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$,

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \Lambda^2\mathfrak{g} \otimes \mathfrak{g}) = \begin{cases} 4 & \text{if } n \geq 3 \\ 1 & \text{if } n = 2 \end{cases} \quad \text{and} \quad \dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g} \otimes \odot^2 \mathfrak{g}) = 1, \quad \forall n \geq 2.$$

Corollary 5.4. *For* $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ *with* $n \geq 3$,

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \ker \Phi) \geq 3.$$

In fact, using tensors, we have checked that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \ker \Phi)$ is three-dimensional and within it there is a two-dimensional subspace $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \ker \Phi \cap \ker \Psi)$ where Ψ is the composition

$$\Lambda^2\mathfrak{g} \otimes \mathfrak{g} \hookrightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\mathrm{Id} \otimes \langle\_,\_\rangle} \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}.$$

Any homomorphism in $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \ker \Phi) \setminus \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \ker \Phi \cap \ker \Psi)$ will suffice for deriving the critical value of λ as in our proof of Theorem 4.1. This critical value of λ is also obtained by Braverman and Joseph [2, Sections 7.4 and 7.7]. They also remark [2, Section 5.4] that the symplectic case may be dealt with by an ‘extremely rare’ but ‘simple-minded procedure’ going back to Dirac. From the tensorial point of view, the reason for this is that if one naïvely extends a tensor

$$S \in \left(\Lambda^2\mathfrak{sp}(2m, \mathbb{C}) \otimes \mathfrak{sp}(2m, \mathbb{C})\right) \cap \left(\mathfrak{sp}(2m, \mathbb{C}) \otimes \odot^2 \mathfrak{sp}(2m, \mathbb{C})\right)$$

by adding zero components then one obtains a tensor in

$$\left(\Lambda^2\mathfrak{sp}(2n, \mathbb{C}) \otimes \mathfrak{sp}(2n, \mathbb{C})\right) \cap \left(\mathfrak{sp}(2n, \mathbb{C}) \otimes \odot^2 \mathfrak{sp}(2n, \mathbb{C})\right)$$

for any $n > m$. In effect, Braverman and Joseph use this observation and an explicit tensor for the case $n = 2$ to obtain the general case.

Usually, Theorem 1.1 is stated in terms of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. To do this, notice that the generators (1.1) of I_λ may be split into skew and symmetric parts:

$$X \otimes Y - Y \otimes X - [X, Y] \quad \text{and} \quad X \otimes Y + Y \otimes X - 2X \odot Y - 2\lambda\langle X, Y\rangle$$

and that we may define the algebra A_λ in two steps, firstly taking the quotient of the tensor algebra by the skew generators. This gives $\mathfrak{U}(\mathfrak{g})$ and an image ideal $\bar{I}_\lambda$ so that $A_\lambda = \mathfrak{U}(\mathfrak{g})/\bar{I}_\lambda$. What we have shown more precisely in Sections 2–4 is the following:

Theorem 5.5. *For the classical complex simple algebras*

$$\mathfrak{g} = \mathfrak{so}(n, \mathbb{C}), \quad n \geq 5, \qquad \lambda \neq -\frac{n-4}{4(n-1)(n-2)}$$

$$\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}), \quad n \geq 2, \qquad \lambda \neq -\frac{1}{16(n+1)}$$

$$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}), \quad n \geq 3, \qquad \lambda \neq -\frac{1}{8(n+1)}$$

the ideal $\bar{I}_\lambda$ *coincides with* $\mathfrak{U}(\mathfrak{g})$ *if* $\lambda \neq 0$ *whilst* $\bar{I}_0 = \mathfrak{U}_+(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})$, *the unique maximal ideal consisting of elements without constant part.*

Proof. Theorems 2.1, 3.1 and 4.1 say that, in these circumstances, the ideal I_λ contains $\mathfrak{g}$ and hence contains $\bigoplus_{s\geq 1} \bigotimes^s \mathfrak{g}$, whose image is $\mathfrak{U}_+(\mathfrak{g})$ by definition. The conclusions are now immediate from (1.1). □

In all other cases the algebra A_λ is, in fact, infinite-dimensional. For the orthogonal algebras, for example, there are linear differential operators

$$\mathcal{D}_X \quad \text{for all } X \in \circledcirc^s \mathfrak{so}(m+1,1)$$

constructed in [3] that satisfy

$$\mathcal{D}_X\mathcal{D}_Y = \mathcal{D}_{X\circledcirc Y} + \frac{1}{2}\mathcal{D}_{[X,Y]} - \frac{m-2}{4m(m+1)}\mathcal{D}_{\langle X,Y\rangle}, \quad \forall X, Y \in \mathfrak{so}(m+1,1).$$

The corresponding holomorphic differential operators provide a realisation of A_λ for $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$ and $\lambda = -\frac{(n-4)}{4(n-1)(n-2)}$. There are similar linear holomorphic differential operators for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ constructed as follows. Recall that in Section 4 we identified $\mathfrak{sl}(n,\mathbb{C})$ with trace-free tensors $X^a{}_b$. More generally,

$$\circledcirc^s \mathfrak{sl}(n,\mathbb{C}) = \left\{ \underbrace{X^{a}{}_{b}{}^{c}{}_{d}{}^{\cdots}{}_{\cdots}{}^{e}{}_{f}}_{2s \text{ indices}} \quad \text{s.t.} \quad \begin{array}{l} X^{a}{}_{b}{}^{c}{}_{d}{}^{\cdots}{}_{\cdots}{}^{e}{}_{f} = X^{(a}{}_{(b}{}^{c}{}_{d}{}^{\cdots}{}_{\cdots}{}^{e)}{}_{f)} \\ X^{a}{}_{b}{}^{c}{}_{d}{}^{\cdots}{}_{\cdots}{}^{e}{}_{f} \text{ is totally trace-free.} \end{array} \right\}$$

and we define

$$\mathcal{D}_X \equiv (-1)^s X^{a}{}_{b}{}^{c}{}_{d}{}^{\cdots}{}_{\cdots}{}^{e}{}_{f} Z^b Z^d \cdots Z^f \frac{\partial^s}{\partial Z^a \partial Z^c \cdots \partial Z^e}$$

as a holomorphic differential operator acting on $\mathbb{C}^n$. For $X^a{}_b, Y^c{}_d \in \mathfrak{sl}(n,\mathbb{C})$,

$$\mathcal{D}_X\mathcal{D}_Y - \mathcal{D}_Y\mathcal{D}_X = (Y^a{}_c X^c{}_b - X^a{}_c Y^c{}_b) Z^b \frac{\partial}{\partial Z^a} = -[X,Y]^a{}_b Z^b \frac{\partial}{\partial Z^a} = \mathcal{D}_{[X,Y]}$$

and

$$\mathcal{D}_X\mathcal{D}_Y + \mathcal{D}_Y\mathcal{D}_X = 2X^{(a}{}_{(b} Y^{c)}{}_{d)} Z^b Z^d \frac{\partial^2}{\partial Z^a \partial Z^c} + (X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a}.$$

However, if we write

$$X^{(a}{}_{(b} Y^{c)}{}_{d)} = C^{ac}{}_{bd} + D^{(a}{}_{(b}\delta^{c)}{}_{d)} + E\delta^{(a}{}_{(b}\delta^{c)}{}_{d)},$$

where

$$D^a{}_b = \frac{1}{n+2}(X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) - \frac{2}{n(n+2)} X^c{}_d Y^d{}_c \delta^a{}_b \quad \text{and} \quad E = \frac{1}{n(n+1)} X^c{}_d Y^d{}_c,$$

then $C^{ac}{}_{bd}$ and $D^a{}_b$ are trace-free. In particular, $C^{ac}{}_{bd} = (X \circledcirc Y)^{ac}{}_{bd}$ and

$$\begin{aligned} \mathcal{D}_X\mathcal{D}_Y + \mathcal{D}_Y\mathcal{D}_X &= 2(C^{ac}{}_{bd} + D^c{}_b\delta^a{}_d + E\delta^c{}_b\delta^a{}_d) Z^b Z^d \frac{\partial^2}{\partial Z^a \partial Z^c} + (X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a} \\ &= 2\mathcal{D}_{X\circledcirc Y} + 2D^a{}_b Z^b Z^c \frac{\partial^2}{\partial Z^c \partial Z^a} + 2E Z^b Z^c \frac{\partial^2}{\partial Z^c \partial Z^b} + (X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a}. \end{aligned}$$

Now, let us restrict the action of these differential operators to germs ϕ of holomorphic functions defined near some basepoint in $\mathbb{C}^n \setminus \{0\}$ and 'homogeneous of degree w' in the sense that $Z^a\partial/\partial Z^a\phi = w\phi$. We find that

$$\begin{aligned} \mathcal{D}_X\mathcal{D}_Y + \mathcal{D}_Y\mathcal{D}_X &= 2\mathcal{D}_{X\circledcirc Y} + \left(2\frac{w-1}{n+2} + 1\right)(X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a} \\ &\quad + 2\left(\frac{1}{n(n+1)} - \frac{2}{n(n+2)}\right) w(w-1) X^c{}_d Y^d{}_c \\ &= 2\mathcal{D}_{X\circledcirc Y} + \frac{2w+n}{n+2}(X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a} - 2\frac{w(w-1)}{(n+1)(n+2)} X^c{}_d Y^d{}_c. \end{aligned}$$

Assembling these computations we conclude that

$$\mathcal{D}_X\mathcal{D}_Y = \mathcal{D}_{X\circledcirc Y} + \frac{2w+n}{2(n+2)}(X^a{}_cY^c{}_b + Y^a{}_cX^c{}_b)Z^b\frac{\partial}{\partial Z^a} + \frac{1}{2}\mathcal{D}_{[X,Y]} - \frac{w(w-1)}{2n(n+1)(n+2)}\mathcal{D}_{\langle X,Y\rangle}. \tag{5.1}$$

In particular, for $w = -n/2$ we obtain

$$\mathcal{D}_X\mathcal{D}_Y = \mathcal{D}_{X\circledcirc Y} + \frac{1}{2}\mathcal{D}_{[X,Y]} - \frac{1}{8(n+1)}\mathcal{D}_{\langle X,Y\rangle}.$$

These operators provide a realisation of A_λ for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ and $\lambda = -\frac{1}{8(n+1)}$ and, in particular, show that this algebra is infinite-dimensional. When w is an integer, the operators $\mathcal{D}_X$ are better regarded as acting between homogeneous line bundles on $\mathbb{CP}_{n-1}$ and it is interesting to note that Fox [5] has shown that (5.1) has a 'curved analogue' valid in any projective differential geometry (whether or not w is integral).

If $n = 2$ we can proceed further because, in this case,

$$X^a{}_cY^c{}_b + Y^a{}_cX^c{}_b = X^d{}_cY^c{}_d\delta^a{}_b = \frac{1}{4}\langle X,Y\rangle\delta^a{}_b$$

whence

$$\mathcal{D}_X\mathcal{D}_Y = \mathcal{D}_{X\circledcirc Y} + \frac{1}{2}\mathcal{D}_{[X,Y]} + \frac{w(w+2)}{24}\mathcal{D}_{\langle X,Y\rangle}$$

for any $w \in \mathbb{C}$. In particular, this shows that A_λ is infinite-dimensional for $\mathfrak{sl}(2,\mathbb{C})$ no matter what λ is.

An alternative to these geometric realisations of A_λ is provided by the generalised Poincaré–Birkhoff–Witt Theorem of Braverman and Gaitsgory [1], which enables one to identify the associated graded algebra $\mathrm{gr}(A_\lambda)$. Specifically, if we let $R \subset \mathfrak{g}\otimes\mathfrak{g}$ be the $\mathfrak{g}$-invariant complement to $\mathfrak{g}\circledcirc\mathfrak{g}$ and $J(R)$ be the two-sided ideal in $\bigotimes\mathfrak{g}$ generated by R, then as a special case of [1] we obtain criteria under which the canonical surjection $p : \bigotimes\mathfrak{g}/J(R) \to \mathrm{gr}(A_\lambda)$ of graded algebras is an isomorphism. These criteria are then verified by Braverman and Joseph [2] in the case of critical λ. It follows from a result of Kostant (given in a lecture at MIT in 1980 and explained with proof in [6, Chapter 3]) that the graded algebra $\bigotimes\mathfrak{g}/J(R)$ is simply the Cartan algebra $\circledcirc\mathfrak{g} = \bigoplus_{s=0}^{\infty}\circledcirc^s\mathfrak{g}$ for any complex simple Lie algebra $\mathfrak{g}$. This is also proved by tensorial means in [3] for the orthogonal algebras, in [4] for the special linear algebras, and the symplectic algebras are easily dealt with by a similar argument. In [4], however, it was incorrectly asserted that p is always an isomorphism.

Acknowledgments

This research was supported by the Australian Research Council, the grants GAČR 201/03/P137, GAČR 201/05/2117, and by the research grant LC505. The first author would also like to thank the Eduard Čech Center and the Mathematical Institute of Charles University for their excellent hospitality during the preparation of this article.

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